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COMMENTS ON A RECENT PAPER BY W. ZIJL

H. S. GREENSIDE

Department of Computer Science and Department of Physics, Duke University, Durham, NC 27706, U.S.A.

SUMMARY

A recent paper by W. Zijl, which reformulated the Navier–Stokes and Boussinesq equations in terms of Clebsch potentials, has an error that greatly reduces the generality of the results. Some other recent efforts to use such potentials in fluid and plasma dynamics are briefly discussed.

KEY WORDS Incompressible flow Clebsch potentials Three-dimensional

In a recent paper of this journal, W. Zijl presented an alternative formulation of the Navier–Stokes and Boussinesq equations in terms of generalized potentials.[1] This formulation was of interest since numerous researchers have tried unsuccessfully to generalize the two-dimensional stream-function to three-dimensional flows. As Zijl correctly points out, there are difficulties associated with primitive variable or vorticity formulations of the fluid equations with non-periodic boundary conditions.

Unfortunately, there is an error in Zijl's analysis that greatly reduces the generality of his formulation. Furthermore, there is no simple way to correct the consequences of this error, so that the usefulness of generalized potentials for numerical simulation remains unresolved. Some attempts by other researchers to express the three-dimensional fluid equations in terms of potentials are discussed towards the end of this comment.

Zijl's approach was to introduce Clebsch potentials $m(x, y, z, t)$, $\psi(x, y, z, t)$ and $\phi(x, y, z, t)$ for the vorticity and velocity fields,

$$\nabla \times \mathbf{v} = \nabla m \times \nabla \psi, \tag{1}$$

$$\mathbf{v} = \nabla \phi + m \nabla \psi, \tag{2}$$

and then to resolve the fluid equations along the basis vectors ∇m, $\nabla \psi$ and $\nabla m \times \nabla \psi$. Instead of solving for the components of the velocity field, one then solves for the three potentials with appropriate boundary conditions. An advantage of this approach is that the velocity is explicitly divergence-free, while natural boundary conditions on m and ψ can be found.

A crucial but incorrect step in Zijl's discussion was the expression for the divergence of the fluid stress tensor

$$\mathbf{S} = \mu[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}] + (k - \tfrac{1}{2}\mu)(\nabla \cdot \mathbf{v})\mathbf{I} \tag{3}$$

in terms of the potentials

$$\nabla \cdot \mathbf{S} = \mu(\nabla^2 m + b)\nabla \psi - \mu(\nabla^2 \psi + \beta)\nabla m + \nabla[(k + \tfrac{4}{3}\mu)(\nabla \cdot \mathbf{v})]. \tag{4}$$

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This is equation (6) of Reference 1. The dynamic viscosity μ and the bulk viscosity k are assumed to be constant. The scalar fields b and β are functions of m and ψ, and are given in Zijl's paper.

Equation (4) gives an alternative formulation of the fluid equations. For example, the Boussinesq equation

$$D\mathbf{v}/Dt=\chi\mathbf{g}-\nabla\pi+\nabla\cdot\mathbf{S}/\rho_0$$

becomes

$$\left(\frac{Dm}{Dt}-\nu(\nabla^2 m+b)\right)\nabla\psi-\left(\frac{D\psi}{Dt}-\nu(\nabla^2\psi+b)\right)\nabla m=\chi\mathbf{g}-\nabla\left(\frac{\partial\phi}{\partial t}+m\frac{\partial\psi}{\partial t}+\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}+\frac{p}{\rho_0}-\Omega\right), \quad (5)$$

where $\nu=\mu/\rho_0$ is the kinematic viscosity and $\mathbf{g}=\nabla\Omega$ is the gravitational acceleration. The corresponding equation for the Navier–Stokes equation is obtained by setting $\chi=0$.

Although equation (5) represents a useful separation of variables with important implications for numerical simulations, it is correct only under very special circumstances. The problem is the expression for $\nabla\cdot\mathbf{S}$ in equation (4). This was derived by assuming the vector identity (derived in the Appendix of Zijl's paper)

$$[(\mathbf{a}\cdot\nabla)\mathbf{b}-(\mathbf{b}\cdot\nabla)\mathbf{a}]\cdot(\mathbf{a}\times\mathbf{b})=0; \quad (6)$$

the special case $\mathbf{a}=\nabla m$ and $\mathbf{b}=\nabla\psi$ leads to equation (4). This identity states that the expression in square brackets generally lies in the linear span of the vector fields $\mathbf{a}$ and $\mathbf{b}$.

Equation (6) does not hold generally, which implies that the right side of equation (4) should have an additional term of the form $h\nabla m\times\nabla\psi$ for some scalar field h. The presence of this new term completely changes Zijl's analysis, e.g. the curl of the right side of equation (5) no longer vanishes, and a separation of the fluid equations into separate equations for m and ψ no longer is possible. The author's reformulation of the fluid equations is therefore useful only under the extremely restrictive condition that the potentials m and ψ satisfy

$$[(\nabla m\cdot\nabla)\nabla\psi-(\nabla\psi\cdot\nabla)\nabla m]\cdot(\nabla m\times\nabla\psi)=0. \quad (7)$$

To see that both equation (7) and equation (6) do not generally hold, consider the counter-example given by $m=x$ and $\psi=y+xz$:

$$\mathbf{a}(x,y,z)=\nabla m=\hat{\mathbf{x}},$$
$$\mathbf{b}(x,y,z)=\nabla\psi=\hat{\mathbf{y}}+x\hat{\mathbf{z}}+z\hat{\mathbf{x}}.$$

Then $(\mathbf{a}\cdot\nabla)\mathbf{b}-(\mathbf{b}\cdot\nabla)\mathbf{a}=\hat{\mathbf{z}}$ and $\mathbf{a}\times\mathbf{b}=\hat{\mathbf{z}}-x\hat{\mathbf{y}}$, so that the left side of equation (6) does not vanish anywhere.

The numerical example in Section 8 of Reference 1, concerning two-dimensional steady Stokes flow in a rectangular driven cavity, turns out to be a special case in which equation (7) is obeyed. It does not generalize to three-dimensional flows and is therefore misleading as to the utility of using potentials.

It is interesting to observe that even for restricted flows such that the potentials satisfy equation (7), Zijl still did not succeed in reformulating the fluid equations in their most general form. Setting the curl of equation (5) to zero leads to an equation of the form:

$$\nabla\alpha\times\nabla\psi+\nabla\beta\times\nabla m=0$$

for scalar fields α and β. The most general solution consistent with gauge invariance is

$$\alpha=f(\psi)+c_1\,\partial_\psi h(m,\psi),$$
$$\beta=g(m)-c_1\,\partial_m h(m,\psi)$$

for arbitrary functions $f(\psi)$, $g(m)$, and $h(m, \psi)$. Zijl makes unnecessary and incorrect restrictions that f, g and h are only functions of time in deriving his equations (10a) and (10b). Not all solutions of the fluid equations satisfy his formulation in terms of potentials.

Another subtlety not mentioned by the author, but one that is familiar to researchers involved with fusion plasma research, is that the generalized potentials are no longer single-valued for domains that are not simply connected,[2] which complicates their use in numerical simulation. In toroidal domains specified by two periodic angles θ and ϕ, any divergence-free field can be represented in a special form in terms of two single-valued potentials ψ and χ:

$$\mathbf{v} = \nabla\psi \times \nabla\theta + \nabla\phi \times \nabla\chi.$$

Although this representation has led to useful numerical advances in toroidal plasma simulation,[3] the utility of such an expression in non-toroidal domains has not yet been demonstrated.

Other researchers, not mentioned by Zijl, have also explored the use of potentials in simplifying fluid equations. Chang has given both theoretical and numerical examples of the use of potentials for Euler flow in a turning channel,[4] while Murdock has discussed ways to implement vorticity potential methods for the three-dimensional Navier–Stokes equations.[5] Busse, in numerous papers, has emphasized that a poloidal–toroidal potential decomposition of the velocity

$$\mathbf{v} = \nabla \times (\psi\hat{\mathbf{z}}) + \nabla \times \nabla \times (\phi\hat{\mathbf{z}})$$

provides a natural way to treat incompressibility for the Boussinesq equations.[6] Although important for theoretical analysis, this leads to PDEs of high spatial order that are awkward to solve numerically except when two of the three spatial variables are periodic.[7] None of these efforts have succeeded in a satisfactory reformulation of the fluid equations that is valid in non-periodic domains with non-slip walls, which remains an open problem.

Finally, it should be pointed out that recent mathematical advances in treating the incompressibility condition in the primitive variable formulation also reduce the need to reformulate the fluid equations in terms of potentials. A summary of several direct methods is given in a recent book by Canuto *et al.*,[8] while a recent discussion of iterative methods is given by Maday and Patera.[9]

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